A theoretical guide to quasiperiodic tilings

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The lecture starts with a general introduction to quasiperiodic tilings, exemplified by the celebrated Penrose tilings in two dimensions [1]. The vertices of a quasiperiodic tiling is supported on a *Z-module* [2], whose rank, i.e., the number of generating bases, is higher than the number of space dimensions. This is in contrast to that the rank of a periodic lattice is given by the number of space dimensions. Four known techniques to generate quasiperiodic tilings, namely, the *cut-and-project*, *section*, *grid*, and *inflation methods* are summarized and compared [3].

Dimensionality and *point group symmetry* are the crudest criteria for classifying quasiperiodic tilings. Down the classification hierarchy to *Bravais classes*, we restrict ourselves to *octagonal*, *decagonal* and *dodecagonal Bravais classes* in two dimensions and *primitive*, *face-centered* and *body-centered icosahedral Bravais classes* in three dimensions. The minimal choice of a Z-module is unique to each of the Bravais classes. Tilings in the same Bravais class would look dissimilar to each other if they have different sets of *prototiles* (i.e., shapes of the tiles). Even so, one of them might be mapped from another just by re-drawing the tile boundaries according to locally determined rules. In such a case, the former tiling is said to be *locally derivable* from the latter. If the two are *mutually locally derivable*, it is said that they belong to the same *mutual local derivability (MLD) class* [4,2]. An atomic decoration of a quasiperiodic tiling can be obtained as an atomic decoration of another tiling in the same MLD class, so that the distinction between tilings in the same MLD class is physically unessential. MLD relationships between different versions of Penrose tilings (P1, P2, P3 and Robinson-triangle tilings) [1] are relatively well documented. Another MLD class in the decagonal family is represented by the Tübingen tiling [2]. MLD classes can be differentiated from each other according to how the window vertices are embedded w.r.t. the Z-module in the perpendicular space [5]. The finest of the classification hierarchy is given by *local isomorphism (LI)*. Two instances of tiling with the same set of prototiles are said to be *locally isomorphic* to each other if an arbitrarily large patch in one of them can be found in the other, and *vice versa* [6]. Locally isomorphic tilings are indistinguishable in arbitrarily large scale as long as the scale is finite. Tilings in the same LI class are related through a parallel shift of the window along the perpendicular space. The MLD relationships above are in fact defined as equivalence relationships between LI classes rather than between tilings [4].

The rest of the lecture will exploit a systematic inflation technique [7] to generate well-known tilings with non-crystallographic symmetries in two and three dimensions. These will cover Ammann-Beenker (octagonal), Penrose / Tübingen (decagonal), Shield / Stampfli (dodecagonal) and the Ammann-Kramer-Neri (icosahedral) tilings, which all have polygonal or polyhedral windows.

References:

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